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LETTER TO THE EDITOR

Cuntz deformations of the exterior algebra

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Abstract. We study here possible deformations of the usual exterior algebra of forms in an N-dimensional space X. For consistency reasons, these deformations are Cuntz derivations on the commutative algebra of functions of X and, moreover, they are solutions of the quantum Yang–Baxter equation. Finally, consistent deformations of the exterior algebra in the $N=1$ and $N=2$ cases are explicitly constructed and the relation of the present approach to the differential calculus on quantum spaces is briefly discussed.

Although non-commutative geometry has deep roots into quantum mechanics, this notion has been introduced by Connes in his extension of the calculus of differential forms and the de Rham homology of currents [1]. Among the first implications was the construction of Yang–Mills–Higgs theory employing the $\mathbb{C}(X)\oplus\mathbb{C}(X)$ and later $(H\oplus\mathbb{C})\otimes\mathbb{C}(X)$ algebra, where $H,\mathbb{C}$ are the fields of quarternions and complex numbers respectively and $\mathbb{C}(X)$ is the algebra of functions in X [2] . Since then, there has existed a growing interest among theorists in studying non-commutative geometry [3,4]. A major reason for this is that non-commutative geometry is ultimately related to quantum groups [5]. The latter are connected with some important aspects of physics, such as quantum spin chains [6], conformal field theories [7], quantum integrable models [8], and so on. However, the most celebrated motivation for such studies is that, possibly, non-commutative geometry will offer a way out of ultraviolet divergencies in quantum field theory [9]. We mention Madore's confrontation of the problem in his fuzzy sphere construction [10]. He observed that if the coordinates of a space are non-commuting operators, then there will exist an uncertainty principle between them. As a result, the space will gain a cellular structure like the one already met in the phase space of quantum mechanics. This will lead, hopefully, to a removal of ultraviolet divergencies and consequently to finite results in field theory. In particular, a space with these properties has been constructed, namely the fuzzy sphere space. This is a sphere embedded in $\mathbb{R}^3$ with coordinates x^μ and SU(2) commutation relations. The corresponding problem for the Euclidean space $\mathbb{R}^N$ has also been considered elsewhere [11].

There also exists another proposal by Dimakis *et al* [12] in which the coordinates are kept commutative but there exists non-commutativity between the coordinates and their differentials. In this particular case the continuity is lost and the space acquires a canonical lattice structure with lattice spacing a where a is the deformation parameter. Thus a deformation of the usual differential calculus produces a totally different space. A natural question that can be addressed is; do other deformations in the differential calculus exist and

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what structure they will produce? We will see that there exist such deformations specified by a Cuntz derivation u [13] which, however, must satisfy the classical Yang–Baxter equation [14].

It should also be noted that there exists another approach in the quantum groups framework. In particular, following Woronowicz [15] and Wess and Zumino [16], one can deal with a differential calculus in a space where neither the coordinates nor the differentials are the conventional ones. Thus, for the quantum plane [17] for example, a consistent calculus exists and some interesting problems can be solved [18].

It will be instructive first to see how one may build a differential calculus for an algebra $\mathcal{A}$. Let us suppose that $\mathcal{A}$ is an algebra with elements a. We associate with every such element a symbol $\mathrm{d}a$ and subsequently, we may construct the universal differential envelope $\Omega\mathcal{A}$ of $\mathcal{A}$ as the space spanned by words build up of a and $\mathrm{d}a$ [1, 3, 19]. In particular, we may express $\Omega\mathcal{A}=\oplus\Omega^p\mathcal{A}$, where $\Omega^p\mathcal{A}$ is the vector space of elements of the form $\omega = a^0\,\mathrm{d}a^1\ldots\mathrm{d}a^p$, $a\in\mathcal{A}$. Moreover, the symbol d regarded as an operator acting on $\Omega\mathcal{A}$, satisfies:

(i) The nilpotency condition

$$\mathrm{d}^2 = 0. \tag{1}$$

(ii) The Leibniz rule

$$\mathrm{d}(\omega_p\omega) = (\mathrm{d}\omega_p)\omega + (-1)^p\omega_p\,\mathrm{d}\omega. \tag{2}$$

In view of (2), every element of $\Omega\mathcal{A}$ can be written as a linear combination of elements of the form

$$a^0 \qquad a^0\,\mathrm{d}a^1\ldots\mathrm{d}a^n \qquad \mathrm{d}a^1\ldots\mathrm{d}a^n. \tag{3}$$

For example, one may easily verify that

$$a^0(\mathrm{d}a^1)a^2 = a^0\,\mathrm{d}(a^1a^2) - (a^0a^1)\mathrm{d}a^2.$$

The above treatment is quite general and states that whenever an algebra $\mathcal{A}$ is given as well as an operation which satisfies equations (1), (2), then we can construct a consistent differential envelope for the algebra.

Let us now recall that there exists a natural endomorphism $1:\mathcal{A}\to\mathcal{A}$ which is specified by $1(a) = a$. Let us define also a second endomorphism p by deforming the previous one so that $p(a) = a - q(a)$, where $q:\mathcal{A}\to\mathcal{A}$. One may then easily verify that, in order for p to be an endomorphism, q must satisfy [3, 13]

$$q(ab) = q(a)b + aq(b) - q(a)q(b). \tag{4}$$

Thus, q is a Cuntz derivation and satisfies a modified Leibniz rule. The above relation may also be written as

$$q(ab) = q(a)b + p(a)q(b) \tag{5}$$

so that one may view q as a derivation twisted by the endomorphism p. We may now proceed along the same lines as before and define the universal object $Q\mathcal{A}$, the Cuntz algebra, which is generated by the $a\in\mathcal{A}$ and the symbol $q(a)$ [13]. In view of the relations (4), (5), every element in the algebra may be written as a linear combination of elements of the form

$$a^0 \qquad a^0 q(a^1)\dots q(a^n) \qquad q(a^1)\dots q(a^n).$$

For example, for the element $a^0 q(a^1) a^2$ we have

$$a^0 q(a^1) a^2 = a^0\,\mathrm{d}(a^1 a^2) - a^0 a^1 q(a^2) - a^0 q(a^1) q(a^2).$$

It should also be noted that there exists a Z_2-construction of the Cuntz algebra carried out by Zekri [20].

To proceed further, let us suppose that $\mathcal{A} = \mathbb{C}(X)$ is the associative algebra of functions on a space X. The differential envelope $\Omega\mathcal{A}$ is then the exterior algebra, that is the algebra of differential forms $\Lambda(X)$ [21]. Thus, $\Omega^0\mathbb{C}(X) = \Lambda^0(X)$ is the vector space of functions on X, $\Omega^1\mathbb{C}(X) = \Lambda^1(X)$ is the space of 1-forms generated by $\mathrm{d}x^i$, $i = 1, 2, \dots, \dim X$, $\Omega^2(X) = \Lambda^2(X)$ is the space of 2-forms generated by $\mathrm{d}x^{i_1}\,\mathrm{d}x^{i_2}$, $i_1 < i_2$ and so on. We may define multiplication of the p-forms $\omega_p \in \Lambda^p$, and the q-form $\omega_q \in \Lambda^q$ to be a $(p+q)$-form $\omega_{p+q} \in \Lambda^{p+q}$. However, in general, their product is not commutative but rather it satisfies

$$\omega_q \omega_p = (-1)^{qp} \omega_p \omega_q$$

and, as a result, the functions, i.e. elements of Λ^0, commute with all forms

$$\omega_p f = f \omega_p.$$

The exterior algebra can be deformed at this point by an ansatz which, in one dimension, may be written as [12]

$$[f, \mathrm{d}x] = a\,\mathrm{d}x\,\partial f \tag{6}$$

where ∂f is the partial derivative of f and a is a parameter. As a consequence of (6) a lattice structure emerges with lattice spacing a and thus, a deformation of the exterior algebra breaks continuity of space. Nevertheless, this is not the most general case one may consider. If we suppose for the commutator $[\Lambda^0(X), \Lambda^1(X)] \subset \Lambda^1(X)$, one may then write, in one dimension, the most general relation

$$[f, \mathrm{d}x] = -\mathrm{d}x\,u(f) \tag{7}$$

where $u(f)$ is a mapping $u : \Lambda^0 \to \Lambda^0$ not specified at the moment.

Let us now try to find the consequences of such relation. Defining the differential of a function f as

$$\mathrm{d}f = \mathrm{d}x\,\partial f \tag{8}$$

(note the relative position of $\mathrm{d}x$ and ∂f) we find for the product fg of two functions that

$$\mathrm{d}(fg) = \mathrm{d}x\,\partial(fg).$$

(Equation (8) defines the so-called right derivative which is different, in the present context, from the left one defined by $\mathrm{d}f = \partial f\,\mathrm{d}x$.) Since the exterior derivative satisfies the Leibniz rule (2), we find, by employing (7) that

$$\mathrm{d}(fg) = \mathrm{d}x\,(\partial f) g + f\,\mathrm{d}x\,(\partial g) = \mathrm{d}x\,(\partial f) g + f \partial g - u(f) \partial g.$$

As a result, one may immediately read off that the derivative satisfies

$$\partial(fg) = (\partial f)g + f\partial g - u(f)\partial g. \tag{9}$$

To find out the nature of the deformation $u(f)$, let us consider the action of the derivative in the triple product fgh. This product can be evaluated in two ways, namely, either as $\partial(fg)h$ or $\partial f(gh)$. In view of the associativity of the algebra, the results must be identical in both cases and thus we will have

$$\partial(fg)h + fg\partial h - u(fg)\partial h = (\partial f)gh + f\partial(gh) - u(f)\partial(gh).$$

Employing above equation (12), we find that the deformation u must satisfy

$$u(fg) = u(f)g + fu(g) - u(f)u(g) \tag{10}$$

which is exactly (4). As a result, a consistent deformation of the usual differential calculus exists if the deformation u is a Cuntz derivation.

Let us now proceed with the exterior algebra in an N-dimensional space X. The exterior derivative d satisfies, as usual, the equations (1), (2) and let us twist the algebra anticipating the deformation $[\Lambda^0, \Lambda^1] \subset \Lambda^1$. Since a base in Λ^1 is the differentials $\mathrm{d}x^i, i = 1, 2, \ldots N$, the anologue of (7) may be written in this case as

$$[f, \mathrm{d}x^i] = -\mathrm{d}x^j\, u_i^i(f) \tag{11}$$

where u_j^i are deformations which must be specified. Proceeding as in the one-dimensional case before, we find that the partial derivative, defined as

$$\mathrm{d}f = \mathrm{d}x^i\, \partial_i f$$

so that

$$(\partial_i x^j) = \delta_i^j$$

satisfies the relation

$$\partial_i(fg) = (\partial_i f)g + f\partial_i g - u_i^j(f)\partial_j g. \tag{12}$$

Employing the associativity of the algebra of functions, the deformations u_i^j must satisfy

$$u_i^j(fg) = u_i^j(f)g + fu_i^j(g) - u_i^k(f)u_k^j(g) \tag{13}$$

which states that $u_i^j(f)$ will also be Cuntz derivations.

We may regard (12) as an operation equation and thus, it can be written as

$$[\partial_i, f] = \partial_i f - u_i^j(f)\partial_j. \tag{14}$$

One may also expect a non-trivial action of the partial derivative ∂_i on the differentials $\mathrm{d}x^j$. We may express that as

$$[\partial_i, \mathrm{d}x^j] = \mathrm{d}x^l\, v_l^j(\partial_i). \tag{15}$$

However, the v_l^j are not expected to be independent of u_l^i. To find out their relation, let us evaluate the quantity $\partial_i f\,\mathrm{d}x^j$. One may easily verify, in view of (11) that

$$\partial_i(f\,\mathrm{d}x^j) = \partial_i f\,\mathrm{d}x^j = \mathrm{d}x^j\,\partial_i f - \mathrm{d}x^k\,u_k^j(\partial_i f).$$

The same quantity may also be evaluated after interchanging f and $\mathrm{d}x^j$ and using (14). In this case we find that

$$\partial_i(f\,\mathrm{d}x^j) = \partial_i(\mathrm{d}x^j\,f - \mathrm{d}x^k\,u_k^j(f)) = \mathrm{d}x^l\,v_l^j(\partial_i)f - \mathrm{d}x^l\,v_l^k(\partial_i)u_k^j(f)$$

and, as a result, a relation for v_i^j, and u_i^j may be written as

$$\partial_i u_l^j(f) - v_l^j(\partial_i)f + v_l^k(\partial_i)u_k^j(f) = 0. \tag{16}$$

We are now in a position to compute more complicated expressions like $(\partial f\,\mathrm{d}x^j)g$. Since the operator in front of g contains non-commuting objects, there are two ways to evaluate this expression. Let us first evaluate it as it stands. Then, taking into account equations (14), (17), (18), we find that

$$\begin{aligned}(\partial_i f\,\mathrm{d}x^j)g = {} & \mathrm{d}x^j\,(\partial_i f)g - \mathrm{d}x^l\,u_l^j(\partial_i f)g + \mathrm{d}x^l\,f v_l^j(\partial_i)g \\ & + \mathrm{d}x^j\,f\partial_i g - \mathrm{d}x^l\,u_l^j(f)\partial_i g - \mathrm{d}x^l\,u_l^n(f)u_n^j(\partial_i)g \\ & - \mathrm{d}x^l\,u_i^k(f)v_l^j(\partial_k)g - \mathrm{d}x^j\,u_i^k(f)\partial_k g + \mathrm{d}x^l\,u_l^j(u_i^k(f))\partial_k g \\ & + \mathrm{d}x^l\,u_l^n(u_i^k(f))v_n^j(\partial_k)g.\end{aligned} \tag{17}$$

We can now interchange the position of f and $\mathrm{d}x^j$ and then proceed in the evaluation of the above quantity. The result we get in this way is

$$\begin{aligned}(\partial_i f\,\mathrm{d}x^j)g &= (\partial_i\,\mathrm{d}x^j\,f)g - (\partial_i\,\mathrm{d}x^l\,u_l^j(f))g \\ &= \mathrm{d}x^l\,v_l^j(\partial_i)fg - \mathrm{d}x^l\,v_l^n(\partial_i)u_n^j(f)g + \mathrm{d}x^j\,(\partial_i f)g - \mathrm{d}x^l\,\partial_i u_l^j(f)g.\end{aligned}$$

Thus, we have two different expressions for the same quantity and consistency requirements leads to the constraint

$$\begin{aligned}& f v_l^j(\partial_i)g - u_l^n(f)v_n^j(\partial_i)g - u_i^k(f)v_l^j(\partial_k)g + u_l^n(u_i^k(f))v_n^j(\partial_k)g \\ &\quad = -\delta_l^j\partial_i fg + u_l^j(\partial_i f) + v_l^j(\partial_i)fg - v_l^n(\partial_i)u_n^j(f)g.\end{aligned} \tag{18}$$

The above expression is too complicated to deal with. For this reason, we impose some conditions on the deformations u_i^j and v_i^j and the simplest condition is linearity. Thus we require

$$v_l^j(\partial_k) = v_{lk}^{ji}\partial_i$$

as well as

$$\partial_i u_j^k(f) = u_{ji}^{kl}\partial_l f.$$

If we apply the latter condition to the coordinates x^n, we find that

$$\partial_i u^j_k(x^n) = u^{jn}_{ki}$$

which can be integrated and the result is

$$u^j_k(x^n) = u^{jn}_{ki} x^i + C^{jn}_k. \tag{19}$$

In this particular case, the condition (19) may be written as

$$v^{jk}_{li} - u^{kj}_{li} - v^{mn}_{li} u^{kj}_{mn} = 0 \tag{20}$$

or, defining

$$V^{ij}_{kl} = \delta^i_k \delta^j_l + v^{ij}_{kl} \tag{21}$$

$$U^{ij}_{kl} = \delta^i_l \delta^j_k - u^{jk}_{kl} \tag{22}$$

we may write (20) as

$$V^{ij}_{mn} U^{kl}_{ij} = \delta^l_m \delta^k_n. \tag{23}$$

In the notation of (22), (21), equation (18) has the simple expression

$$V^{lb}_{im} U^{jk}_{la} U^{ac}_{bd} = U^{lk}_{ib} V^{jc}_{la} U^{ab}_{md} \tag{24}$$

$$V^{lb}_{im} U^{jk}_{ld} C^{ad}_{b} = U^{lk}_{ib} V^{ja}_{ln} C^{bn}_{m}. \tag{25}$$

We can use (23) so that the above relations may be expressed as

$$U^{lb}_{im} U^{jk}_{la} U^{ac}_{bd} = U^{jl}_{ia} U^{ac}_{md} U^{kc}_{lb} \tag{26}$$

$$U^{lb}_{im} U^{jk}_{la} C^{ca}_{b} = U^{jl}_{ia} U^{ac}_{md} C^{kc}_{l} \tag{27}$$

which is just the quantum Yang–Baxter equations (QYBE) [14] in component form. As a result, the consistency of non-standard calculus is ultimately related to the existence of solutions of the Yang–Baxter system (26), (27).

Having established the general framework, we are now in a position to apply our findings to some particular cases. In particular, we will focus our attention on the $N = 1$ and $N = 2$ cases and we will try to find exotic calculus on the one- and two-dimensional Euclidean space $\mathbb{R}^1$ and $\mathbb{R}^2$ respectively.

To begin with, let us first consider the $N = 1$ case. Here the functions are the usual functions on $\mathbb{R}^1$ (parametrized by x) and (10) for $f = x$ is written as

$$[x, \mathrm{d}x] = a\,\mathrm{d}x\,x + b\,\mathrm{d}x \tag{28}$$

where we have imposed the linearity condition (19)

$$u(x) = -ax - b.$$

We can also write (28) as

$$x\,\mathrm{d}x = c\,\mathrm{d}x\,x + b\,\mathrm{d}x$$

with $c = 1 + a$. One may easily verify that

$$x^n\,\mathrm{d}x = \mathrm{d}x\,(cx + b)^n$$

so that

$$f(x)\,\mathrm{d}x = \mathrm{d}x\,f(cx + b).$$

As a result, (9) may be written in this case as

$$\partial(fg)(x) = \partial f(x)g(x) + f(cx + b)\partial g(x)$$

which leads to the expression

$$\partial f(x) = \frac{f(cx + b) - f(x)}{x(c - 1) + b} \tag{29}$$

for the partial derivative, where the condition $\partial x = 1$ has been imposed. It is interesting to note that if $c = 1$, then (29) is just the discrete derivative [12], while when $b = 0$, (29) is the so called q-derivative [25]. This indicates a relation between the Cuntz deformations and q-bosons [22]. Indeed, if we consider ∂x as an operator acting on functions, then it follows that

$$\partial x = 1 + cx\partial.$$

This relation in the Fock–Bargmann representation ($\alpha \to \partial, \alpha^\dagger \to x$), is expressed as

$$\alpha\alpha^\dagger - c\alpha^\dagger\alpha = 1$$

which is the commutation relation for q-particles obeying infinite statistics [23].

Before proceeding with the $N = 2$ case, some remarks are in order. As we have seen, the deformation parameter must be a solution of the QYBE in order to have a consistent calculus. However, some of these solutions may not be appropriate because we must also respect the usual commutativity of functions on $\mathbb{R}^2$. This commutativity introduces constraints which can be formulated as follows. The deformation parameters satisfy (13) and if we interchange the function f with g the result will be the same since the algebra is commutative. This observation leads to the condition

$$u^j_i(f)u^k_j(g) = u^j_i(g)u^k_j(f). \tag{30}$$

If we define the matrices

$$U^l = u^j_i(x^l)$$

then (30) may be written as

$$[U^m, U^n] = 0. \tag{31}$$

As a result, exotic calculus in the usual exterior algebra exist if in addition to equations (26) and (27), (31) also holds.

Let us now turn to the $N = 2$ case. Here, all the solutions of the constant QYBE are known [24]. So it is straightforward to verify that there exist two solutions which satisfy all the consistency requirements, namely equations (26), (27) and (31). These solutions can be written in a matrix form as

$$U_1 = (U^{ij}_{kl}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s & 0 & 0 & 1 \end{pmatrix} \qquad U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s & 0 & 0 & -1 \end{pmatrix} \tag{32}$$

where the upper indices count the rows and the lower the columns in the order (11,12,21,22).

In analogy, the V^{kl}_{ij} may be expressed as

$$V_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ s^{-1} & 0 & 0 & 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ s^{-1} & 0 & 0 & -1 \end{pmatrix}. \tag{33}$$

Finally, to specify the constants C^{ji}_{k}, we observe that choosing

$$C^{ji}_{k} = U^{ji}_{kl}\lambda^l$$

with λ^l a constant vector, (25) or (27) is automatically satisfied in view of (24) or (26) respectively.

As a result, the exotic calculus in the $N = 2$ case can be expressed analytically as

$$\begin{aligned} x\,\mathrm{d}x &= \mathrm{d}x\,x + s\,\mathrm{d}y\,y + \lambda\,\mathrm{d}y \\ y\,\mathrm{d}x &= \mathrm{d}x\,y \\ x\,\mathrm{d}y &= \mathrm{d}y\,x \\ y\,\mathrm{d}y &= \mathrm{d}y\,y \end{aligned} \tag{34}$$

which corresponds to the U_1 solution and

$$\begin{aligned} x\,\mathrm{d}x &= \mathrm{d}x\,x + s\,\mathrm{d}y\,y + \lambda\,\mathrm{d}y \\ y\,\mathrm{d}x &= \mathrm{d}x\,y \\ x\,\mathrm{d}y &= \mathrm{d}y\,x \\ y\,\mathrm{d}y &= -\mathrm{d}y\,y \end{aligned} \tag{35}$$

which corresponds to the U_2 solution. It should be noted that one may relax the condition of the commutativity of the functions. In this case there also exists a consistent differential calculus such as the well established differential calculus on the quantum plane [16–18].

As becomes clear from the above, the deformations one can make in the usual exterior algebra of forms are not unique. However, they can uniquely be specified by the Cuntz relation (13) and the Yang–Baxter system (26), (27). The solution given in [12] fits in this class of deformations and actually corresponds to the simple solution

$$u_{ij}^{kl} = 0 \qquad C_k^{ij} \neq 0.$$

As a final comment, let us note the similarity of the present approach to the quantum groups framework where analogous results are obtained [15, 16]. The fundamental difference between the two cases is the commutativity of functions in the former case while in the latter the functions do not commute.

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